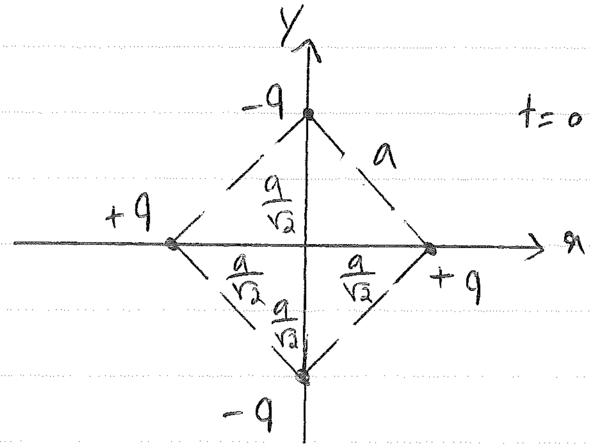
Problem Session 11 04/10/2019

Problem 9.2, Jackson.



The position of the four charges as a function of time follows:

$+q$ (right): $\left(\frac{a}{\sqrt{2}}\cos\omega t, \frac{a}{\sqrt{2}}\sin\omega t\right)$

$+q$ (left): $\left(-\frac{a}{\sqrt{2}}\cos\omega t, -\frac{a}{\sqrt{2}}\sin\omega t\right)$

$-q$ (top): $\left(\frac{-a}{\sqrt{2}}\sin\omega t, \frac{a}{\sqrt{2}}\cos\omega t\right)$

$-q$ (bottom): $\left(\frac{a}{\sqrt{2}}\sin\omega t, \frac{-a}{\sqrt{2}}\cos\omega t\right)$

It is easy to see that the electric dipole moment and the magnetic dipole moment of the configuration is zero at all times. Therefore, the first non-vanishing contribution is due to the electric quadrupole moment. We have:

$$(Q_{11})_{phys} = \int (3x^2 - r^2)\, \rho(\vec{x}, t)\, d^3x = \sum_{i=1}^{4} q_i (3x_i^2 - r_i^2) = 3qa^2 \cos(2\omega t)$$

$$= Re\,(3qa^2 e^{-2i\omega t}) \Rightarrow \boxed{Q_{11} = 3qa^2}$$

$$(Q_{22})_{phys} = \int (3y^2 - r^2)\, \rho(\vec{x}, t)\, d^3x = \sum_{i=1}^{4} q_i (3y_i^2 - r_i^2) = -3qa^2 \cos(2\omega t)$$

$$= Re\,(-3qa^2 e^{-2i\omega t}) \Rightarrow \boxed{Q_{22} = -3qa^2}$$

$$(Q_{12})_{phys} = (Q_{21})_{phys} = \int 3xy\, \rho(\vec{x}, t)\, d^3x = \sum_{i=1}^{4} 3x_i y_i q_i = -3qa^2 \sin(2\omega t)$$

$$= Re\,(-3qa^2 i\, e^{-2i\omega t}) \Rightarrow \boxed{Q_{12} = Q_{21} = -3qa^2 i}$$

All of the other Q_{ij}'s are zero.

Thus, for the direction vector $\hat{n}$, we have:

$$q_1(\hat{n}) = \sum_j Q_{ij} n_j = 3qa^2 (n_x - i n_y)$$

$$q_2(\hat{n}) = \sum_j Q_{2j} n_j = -3qa^2 (n_x + i n_y) = -i\, q_1(\hat{n})$$

We note that radiation has frequency 2ω.

For the radiated fields, in the long wavelength limit, we have:

$$\vec{H} = \frac{-8ick^3}{24\pi} \frac{e^{2ikr}}{r}\, \hat{n} \times q_1(\hat{n})$$

Note that:

$$\hat{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z} = n_+ \hat{e}_- + n_- \hat{e}_+ + n_z \hat{z}$$

Where:

$$n_{\pm} \equiv \frac{n_x \pm i n_y}{\sqrt{2}} \qquad (\hat{e}_+ \times \hat{e}_- = -i\hat{z},\ \hat{z} \times \hat{e}_{\pm} = \mp i \hat{e}_{\pm})$$

This implies that:

$$\boxed{\vec{H} = \frac{-i\sqrt{2}\, c k^3}{\pi} q a^2 \frac{e^{2ikr}}{r} \left[n_-(-i\hat{z}) + i n_z \hat{e}_- \right]}$$

Also:

$$\vec{E} = \frac{-1}{i2\omega\varepsilon_0} \vec{\nabla} \times \vec{H} = \frac{-2ik}{2i\omega\varepsilon_0} \hat{n} \times \vec{H}$$

This results in:

$$\boxed{\vec{E} = \frac{-i\sqrt{2}\, k^3}{\pi\varepsilon_0} q a^2 \frac{e^{2ikr}}{r} (n_x - i n_y) \left[\frac{1}{2}(n_x^2 + n_y^2 + 2n_z^2)\hat{e}_- - n_- n_z \hat{z} - \frac{1}{2}(n_x^2 - n_y^2 + 2i n_x n_y)\hat{e}_+ \right]}$$

The expressions for $\vec{E}$ (and $\vec{H}$) can be written in terms of $\hat{x}, \hat{y}, \hat{z}$.

The angular distribution of the radiation is given by:

$$\frac{dP}{d\Omega} = \frac{c(2k)^6}{1152\,\varepsilon_0 \pi} |q_1(\hat{n})|^2 \, |(\hat{n} \times (\hat{x} - i\hat{y})) \times \hat{n}|^2$$

But:

$$|(\hat{n} \times (\hat{x} - i\hat{y})) \times \hat{n}| = |(\hat{n}\cdot\hat{n})(\hat{x} - i\hat{y}) - (\hat{n}\cdot(\hat{x} - i\hat{y}))\hat{n}| = |\hat{x} - i\hat{y} -$$

$$(n_x - i n_y)(n_x\hat{x} + n_y\hat{y} + n_z\hat{z})| = |(1 - n_x^2 + i n_x n_y)\hat{x} - (i + n_x n_y - i n_y^2)\hat{y}$$

$$- (n_x - i n_y) n_z \hat{z}| = (1-n_x^2)^2 + n_x^2 n_y^2 + n_x^2 n_y^2 + (1-n_y^2)^2 + (n_x^2+n_y^2)n_z^2$$

$$= 2 - 2(n_x^2+n_y^2) + \underbrace{(n_x^4 + 2n_x^2 n_y^2 + n_y^4)}_{(n_x^2+n_y^2)^2} + (n_x^2+n_y^2)n_z^2 = 2-2(n_x^2+n_y^2)$$

$$+ (n_x^2+n_y^2)\underbrace{(n_x^2+n_y^2+n_z^2)}_{1} = 2 - (n_x^2+n_y^2) = 2-\sin^2\theta$$

Thus:

$$\boxed{\frac{dP}{d\Omega} = \frac{cq^2a^4k^6}{2\pi^2\varepsilon_0}\sin^2\theta\,(2-\sin^2\theta)}$$

The total radiated power is:

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \boxed{\frac{8}{5}\,\frac{cq^2a^4k^6}{\pi\varepsilon_0}}$$

$d\Omega = \sin\theta\, d\theta\, d\phi$